COMPLETE DERIVATION OF 2D SHALLOW-WATER MODEL FROM THE PRIMITIVE EQUATIONS GOVERNING GEOPHYSICAL FLOWS

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ABSTRACT

The fact that the 2D shallow-water model has been used for decades is such a long time that a complete reference on how to derive it from the primitive equations either has likely to become a very rare article or written in a way that is very complicated for the newcomers. Certain physical assumptions and mathematical theorem should be acquired in order to fully understand how the complete 2D shallow-water model is derived which are often being skipped in many recent ocean modelling text books. In this paper, full derivation of the model that consist of momentum conversation in Cartesian coordinates and the continuity equations will be shown in the simplest way to satisfy the curiosity of fresh physical oceanographers.

Keywords: 2D shallow water, geophysical flows, primitive equations.

1. INTRODUCTION

Consider the respective three momentum conservation equations and the continuity equation out of seven primitive equations governing geophysical flows under Boussinesq, Newtonian-Fluid and Reynolds-Averaged assumptions, in Cartesian coordinate as follows [1]

$$\frac{\partial u}{\partial t} + \mathrm{Adv}(u) - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + \mathrm{Adv}(v) + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left(A_h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right) \tag{2}$$

$$\frac{\partial w}{\partial t} + \mathrm{Adv}(w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - \frac{g\rho}{\rho_0} + \frac{\partial}{\partial x}\left(A_h\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_z\frac{\partial w}{\partial z}\right) \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

in which

$$\mathrm{Adv}(\psi) = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} + w\frac{\partial \psi}{\partial z}$$

where (u, v, w) are the three components of velocity, f is the Coriolis parameter, P is the pressure, g is the gravitational acceleration, ρ is the density of the fluid, ρ_0 is the mean density, A_h is the lateral eddy viscosity, A_z is the vertical eddy viscosity and ψ is representing any field variable.

2. METHODOLOGY

There will be four basic steps to derive SWE from Eq. (1-4). Firstly, one needs to incorporate the hydrostatic balance relation then secondly, to specify boundary conditions (BCs) for a water column. The third step is to carry out the depth-averaged integration and finally to apply the BCs within the integration operation [2]. A typical water column definition is shown in Figure 1. The depth-averaged integration points are from bottom $z = z_0$ until water surface $z = z_0 + h$. Note that the total depth $h = h_0 + \eta$ where h_0 is the undisturbed water depth and η is the water elevation [3].

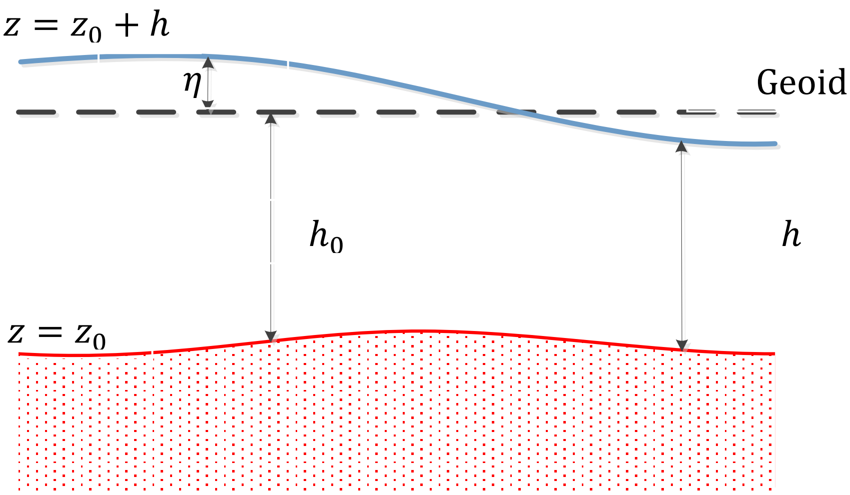

Figure 1 A Typical Water Column

Due to the main condition for shallow-water model is that the horizontal length-scale far exceed the vertical length-scale, many terms in Eq. (3) can be neglected in such it reduces to the simple hydrostatic balance:

$$0 = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - g \tag{5}$$

Integration of Eq. (5) at water surface $z = z_0 + h$, $P = P_a$ atmospheric pressure (assumed constant) [4], yields

$$P(z) = P_a - \rho_0 g z + \rho_0 g(z_0 + h) \tag{6}$$

In SWE another set of variables is being used by the definition of depth-averaged field flow

$$\int_{z_0}^{z_0+h} \psi \, dz = h\bar{\psi} \tag{7}$$

Next, the boundary conditions (BCs) at the surface and at the bottom are defined respectively [5]

$$w_{z_0+h} = \frac{\partial z_{z_0+h}}{\partial t} + u\frac{\partial z_{z_0+h}}{\partial x} + v\frac{\partial z_{z_0+h}}{\partial y} \tag{8}$$

$$w_{z_0} = u\frac{\partial z_{z_0}}{\partial x} + v\frac{\partial z_{z_0}}{\partial y} \tag{9}$$

$$\tau_{z_0+h,x} = -\tau_{xx}\frac{\partial z_{z_0+h}}{\partial x} - \tau_{xy}\frac{\partial z_{z_0+h}}{\partial y} + \tau_{xz} \tag{10}$$

$$\tau_{z_0,x} = \tau_{xx}\frac{\partial z_{z_0}}{\partial x} + \tau_{xy}\frac{\partial z_{z_0}}{\partial y} - \tau_{xz} \tag{11}$$

Beside the BCs, other required tools are Leibnitz integration rule as stated below

$$\frac{\partial}{\partial x}\int_{z_b}^{z_s} u\,dz = \int_{z_b}^{z_s}\frac{\partial u}{\partial x}dz + u|_{z_s}\frac{\partial z_s}{\partial x} - u|_{z_b}\frac{\partial z_b}{\partial x} \tag{12}$$

3. RESULTS AND DISCUSSIONS

Now we can start integrating the derivative and the advection terms of Eq. (1)

$$\begin{aligned}\int_{z_0}^{z_0+h}&\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)dz\\ &= \frac{\partial}{\partial t}\int_{z_0}^{z_0+h} u\,dz + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\,dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\,dz - u|_{z_0+h}\frac{\partial(z_0+h)}{\partial t}\\ &- u^2|_{z_0+h}\frac{\partial(z_0+h)}{\partial x} - (uv)|_{z_0+h}\frac{\partial(z_0+h)}{\partial y} + (uw)|_{z_0+h} + u|_{z_0}\frac{\partial z_0}{\partial t} + u^2|_{z_0}\frac{\partial z_0}{\partial x}\\ &+ (uv)|_{z_0}\frac{\partial z_0}{\partial y} - (uw)|_{z_0}\end{aligned} \tag{13}$$

Equation (13) can be rearrange in the form where the boundary conditions terms become explicit

$$\begin{aligned}\int_{z_0}^{z_0+h}&\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)dz\\ &= \frac{\partial}{\partial t}\int_{z_0}^{z_0+h} u\,dz + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\,dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\,dz\\ &- u|_{z_0+h}\left(\frac{\partial h}{\partial t} + u|_{z_0+h}\frac{\partial(z_0+h)}{\partial x} + v|_{z_0+h}\frac{\partial(z_0+h)}{\partial y} - w|_{z_0+h}\right)\\ &+ u|_{z_0}\left(\frac{\partial z_0}{\partial t} + u|_{z_0}\frac{\partial z_0}{\partial x} + v|_{z_0}\frac{\partial z_0}{\partial y} - w|_{z_0}\right)\end{aligned} \tag{14}$$

Substitute the kinematic BCs, Eq. (8, 9) into Eq. (14) the last two terms can be eliminated and after employing Eq. (7) yields

$$\begin{aligned}\int_{z_0}^{z_0+h}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)dz &= \frac{\partial}{\partial t}\int_{z_0}^{z_0+h} u\,dz + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\,dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\,dz\\ &= \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\,dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\,dz\end{aligned} \tag{15}$$

Integration of the Coriolis term in Eq. (1) is

$$-\int_{z_0}^{z_0+h} fv\,dz = -f\int_{z_0}^{z_0+h} v\,dz = -fh\bar{v} \tag{16}$$

As for the pressure term in Eq. (1), Eq. (6) can be substituted in and then integration can be carried out

$$\begin{aligned} -\int_{z_0}^{z_0+h} \frac{1}{\rho_0}\frac{\partial P}{\partial x} dz &= -\frac{1}{\rho_0}\int_{z_0}^{z_0+h} \frac{\partial}{\partial x}\big(P_a - \rho_0 g z + \rho_0 g(z_0+h)\big)\, dz \\ &= -\frac{1}{\rho_0}\int_{z_0}^{z_0+h} \rho_0 \frac{\partial}{\partial x} g(z_0+h)\, dz = -g\frac{\partial}{\partial x}(z_0+h)\int_{z_0}^{z_0+h} dz = -gh\frac{\partial}{\partial x}(z_0+h) \end{aligned} \tag{17}$$

The last term of Eq. (1) containing the viscous terms, integrating one by one yields

$$\begin{aligned} \int_{z_0}^{z_0+h} \frac{\partial}{\partial x}\left(A_h\frac{\partial u}{\partial x}\right) dz &= \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} A_h\frac{\partial u}{\partial x} dz - A_h\frac{\partial u}{\partial x}\bigg|_{z_0+h}\frac{\partial}{\partial x}(z_0+h) + A_h\frac{\partial u}{\partial x}\bigg|_{z_0}\frac{\partial z_0}{\partial x} \\ &= \frac{\partial}{\partial x}\left(hA_h\overline{\frac{\partial u}{\partial x}}\right) - A_h\frac{\partial u}{\partial x}\bigg|_{z_0+h}\frac{\partial}{\partial x}(z_0+h) + A_h\frac{\partial u}{\partial x}\bigg|_{z_0}\frac{\partial z_0}{\partial x} \end{aligned} \tag{18}$$

$$\begin{aligned} \int_{z_0}^{z_0+h} \frac{\partial}{\partial y}\left(A_h\frac{\partial u}{\partial y}\right) dz &= \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} A_h\frac{\partial u}{\partial y} dz - A_h\frac{\partial u}{\partial y}\bigg|_{z_0+h}\frac{\partial}{\partial y}(z_0+h) + A_h\frac{\partial u}{\partial y}\bigg|_{z_0}\frac{\partial z_0}{\partial y} \\ &= \frac{\partial}{\partial y}\left(hA_h\overline{\frac{\partial u}{\partial y}}\right) - A_h\frac{\partial u}{\partial y}\bigg|_{z_0+h}\frac{\partial}{\partial y}(z_0+h) + A_h\frac{\partial u}{\partial y}\bigg|_{z_0}\frac{\partial z_0}{\partial y} \end{aligned} \tag{19}$$

$$\int_{z_0}^{z_0+h} \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) dz = A_z\frac{\partial u}{\partial z}\bigg|_{z_0+h} - A_z\frac{\partial u}{\partial z}\bigg|_{z_0} \tag{20}$$

Eq. (18-20) are now combined to obtain

$$\begin{aligned} &\frac{\partial(h\bar{u})}{\partial t} + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\, dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\, dz - fh\bar{v} \\ &= -gh\frac{\partial}{\partial x}(z_0+h) + \frac{\partial}{\partial x}\left(hA_h\overline{\frac{\partial u}{\partial x}}\right) + \frac{\partial}{\partial y}\left(hA_h\overline{\frac{\partial u}{\partial y}}\right) \\ &\quad + \left(-A_h\frac{\partial u}{\partial x}\bigg|_{z_0+h}\frac{\partial}{\partial x}(z_0+h) - A_h\frac{\partial u}{\partial y}\bigg|_{z_0+h}\frac{\partial}{\partial y}(z_0+h) + A_z\frac{\partial u}{\partial z}\bigg|_{z_0+h}\right) \\ &\quad + \left(A_h\frac{\partial u}{\partial x}\bigg|_{z_0}\frac{\partial z_0}{\partial x} + A_h\frac{\partial u}{\partial y}\bigg|_{z_0}\frac{\partial z_0}{\partial y} - A_z\frac{\partial u}{\partial z}\bigg|_{z_0}\right) \end{aligned} \tag{21}$$

Implementing the dynamic BCs Eq. (10-11) to the last two terms in the bracket to define lateral stress

$$\begin{aligned} -A_h\frac{\partial u}{\partial x}\bigg|_{z_0+h}\frac{\partial}{\partial x}(z_0+h) - A_h\frac{\partial u}{\partial y}\bigg|_{z_0+h}\frac{\partial}{\partial y}(z_0+h) + A_z\frac{\partial u}{\partial z}\bigg|_{z_0+h} &= \frac{\tau_{\mathrm{wind},x}}{\rho_0} \\ A_h\frac{\partial u}{\partial x}\bigg|_{z_0}\frac{\partial z_0}{\partial x} + A_h\frac{\partial u}{\partial y}\bigg|_{z_0}\frac{\partial z_0}{\partial y} - A_z\frac{\partial u}{\partial z}\bigg|_{z_0} &= -\frac{\tau_{\mathrm{bot},x}}{\rho_0} \end{aligned} \tag{22}$$

So Eq. (21) can now be written as follows

$$\frac{\partial(h\bar{u})}{\partial t} + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u^2\, dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} uv\, dz - fh\bar{v} \tag{23}$$

$$= -gh\frac{\partial}{\partial x}(z_0 + h) + \frac{\partial}{\partial x}\left(hA_h\frac{\overline{\partial u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(hA_h\frac{\overline{\partial u}}{\partial y}\right) + \frac{\tau_{\mathrm{wind},x}}{\rho_0} - \frac{\tau_{\mathrm{bot},x}}{\rho_0}$$

The next treatment is to decompose the momentary vertical field into a depth integrated mean value part and a fluctuation part, in analogy to Reynolds-averaged assumption. For instance, the velocity component in x-axis becomes

$$u(z) = \bar{u} - \tilde{u} \quad \text{with} \int_{z_0}^{z_0+h} \tilde{u}\ dz = 0 \tag{24}$$

where $\bar{u}$ is the mean velocity over the vertical axis and $\tilde{u}$ is the fluctuation part. Additionally, the following rule of integration holds

$$\int_{z_0}^{z_0+h} (\bar{u} + \tilde{u})(\bar{u} + \tilde{u})\ dz = \int_{z_0}^{z_0+h} \bar{u}^2\, dz + \int_{z_0}^{z_0+h} \tilde{u}\tilde{u}\, dz \tag{25}$$

Substitute Eq. (24-25) into Eq. (23) yields the following depth-averaged momentum equation in direction of the x-axis

$$\begin{aligned}&\frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\bar{u}^2)}{\partial x} + \frac{\partial(h\bar{u}\bar{v})}{\partial y} + \frac{\partial}{\partial x}\int_{z_0}^{z_0+h} \tilde{u}\tilde{u}\, dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} \tilde{u}\tilde{v}\, dz - fh\bar{v} \\ &= -gh\frac{\partial}{\partial x}(z_0 + h) + \frac{\partial}{\partial x}\left(hA_h\frac{\overline{\partial u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(hA_h\frac{\overline{\partial u}}{\partial y}\right) + \frac{\tau_{\mathrm{wind},x}}{\rho_0} - \frac{\tau_{\mathrm{bot},x}}{\rho_0}\end{aligned} \tag{26}$$

Analogy to the treatment of Reynolds stresses, through a process called subgrid-scale parameterization, under the assumption of uniform values of eddy viscosity A_h , Eq. (26) can be formulated as

$$\begin{aligned}&\frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\bar{u}^2)}{\partial x} + \frac{\partial(h\bar{u}\bar{v})}{\partial y} - fh\bar{v} \\ &= -gh\frac{\partial}{\partial x}(z_0 + h) + A_h\left[\frac{\partial}{\partial x}\left(h\frac{\overline{\partial u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\overline{\partial u}}{\partial y}\right)\right] + \frac{\tau_{\mathrm{wind},x}}{\rho_0} - \frac{\tau_{\mathrm{bot},x}}{\rho_0}\end{aligned} \tag{27}$$

Similar operation is applied to Eq. (2) to obtain

$$\begin{aligned}&\frac{\partial(h\bar{v})}{\partial t} + \frac{\partial(h\bar{u}\bar{v})}{\partial x} + \frac{\partial(h\bar{v}^2)}{\partial y} - fh\bar{u} \\ &= -gh\frac{\partial}{\partial y}(z_0 + h) + A_h\left[\frac{\partial}{\partial x}\left(h\frac{\overline{\partial v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\overline{\partial v}}{\partial y}\right)\right] + \frac{\tau_{\mathrm{wind},y}}{\rho_0} - \frac{\tau_{\mathrm{bot},y}}{\rho_0}\end{aligned} \tag{28}$$

The depth-averaged integration of the continuity equation Eq. (4) is as follows

$$\begin{aligned}&\int_{z_0}^{z_0+h} \frac{\partial u}{\partial x} dz + \int_{z_0}^{z_0+h} \frac{\partial v}{\partial y} dz + \int_{z_0}^{z_0+h} \frac{\partial w}{\partial z} dz = 0 \\ &\frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u\, dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} v\, dz - u|_{z_0+h}\frac{\partial(z_0 + h)}{\partial x} - v|_{z_0+h}\frac{\partial(z_0 + h)}{\partial y} + w|_{z_0+h} + u|_{z_0}\frac{\partial z_0}{\partial x} \\ &+ v|_{z_0}\frac{\partial z_0}{\partial y} + w|_{z_0} = 0\end{aligned} \tag{29}$$

$$\frac{\partial}{\partial x}\int_{z_0}^{z_0+h} u\,dz + \frac{\partial}{\partial y}\int_{z_0}^{z_0+h} v\,dz + \frac{\partial h}{\partial t} = 0$$
$$\frac{\partial(h\bar{u})}{\partial x} + \frac{\partial(h\bar{v})}{\partial y} + \frac{\partial h}{\partial t} = 0 \tag{30}$$

4. CONCLUSIONS

The results from the depth-averaged operations so far are known as the conservative form of 2D shallow-water equations (SWE) [6], recapitulated as follows

$$\frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\bar{u}^2)}{\partial x} + \frac{\partial(h\bar{u}\bar{v})}{\partial y} - fh\bar{v}$$
$$= -gh\frac{\partial}{\partial x}(z_0 + h) + A_h\left[\frac{\partial}{\partial x}\left(h\frac{\overline{\partial u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\overline{\partial u}}{\partial y}\right)\right] + \frac{\tau_{\text{wind},x}}{\rho_0} - \frac{\tau_{\text{bot},x}}{\rho_0}$$

$$\frac{\partial(h\bar{v})}{\partial t} + \frac{\partial(h\bar{u}\bar{v})}{\partial x} + \frac{\partial(h\bar{v}^2)}{\partial y} - fh\bar{u}$$
$$= -gh\frac{\partial}{\partial y}(z_0 + h) + A_h\left[\frac{\partial}{\partial x}\left(h\frac{\overline{\partial v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\overline{\partial v}}{\partial y}\right)\right] + \frac{\tau_{\text{wind},y}}{\rho_0} - \frac{\tau_{\text{bot},y}}{\rho_0}$$

$$\frac{\partial(h\bar{u})}{\partial x} + \frac{\partial(h\bar{v})}{\partial y} + \frac{\partial h}{\partial t} = 0$$

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